

THE HILBERT SERIES OF A LINEAR SYMPLECTIC CIRCLE QUOTIENT

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ABSTRACT. We study the Hilbert series of the graded algebra of regular functions on a symplectic quotient of a unitary circle representation and elaborate explicit formulas for the lowest coefficients of the Laurent expansion of such a Hilbert series in terms of rational symmetric functions of the weights. Considerable efforts are devoted to including the cases where the weights are degenerate. We find that these Laurent expansions formally resemble Laurent expansions of Hilbert series of graded rings of real invariants of finite subgroups of U_n . Moreover, we prove that certain Laurent coefficients are strictly positive. Experimental observations are presented concerning the behavior of the coefficients, and we provide empirical evidence that these results might generalize to higher dimensional tori and possibly nonabelian groups.

CONTENTS

1. Introduction	1
Acknowledgements	3
2. Background	3
3. The Hilbert series of a symplectic circle quotient	5
3.1. The generic case	5
3.2. The degenerate case	7
3.3. Another approach to the degenerate case: Proof of Theorem 3.1	7
4. An algorithm for computing the Hilbert series	11
5. Laurent expansion of $\text{Hilb}_A^{on}(x)$	13
5.1. The Laurent series coefficients of $H(C)$.	14
5.2. Continuity of the Laurent series coefficients	17
5.3. The completely degenerate case	19
6. Laurent expansion in the case of a finite subgroup of U_n	19
7. Experiments and conjectures	22
Appendix A. Schur polynomials	26
References	27

1. INTRODUCTION

This work is part of a project that attempts to elucidate under which conditions the symplectic quotient M_0 (for details see Section 2) of a unitary representation

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V of a compact Lie group G is regularly symplectomorphic to a quotient \mathbb{C}^n/Γ of a finite subgroup Γ of U_n . The notion of a *regular symplectomorphism* has been introduced in [7]. Roughly speaking, such a regular symplectomorphism is an isomorphism of the Poisson algebras of smooth function $\mathcal{C}^\infty(M_0) \rightarrow \mathcal{C}^\infty(\mathbb{C}^n/\Gamma)$ that comes from lifting an isomorphism of the Poisson algebras of regular functions $\mathbb{R}[M_0] \rightarrow \mathbb{R}[\mathbb{C}^n/\Gamma]$ that satisfies suitable semi-algebraic conditions (cf. [7, Subsection 4.2]). So if the Poisson algebras $\mathbb{R}[M_0]$ and $\mathbb{R}[\mathbb{C}^n/\Gamma]$ are non-isomorphic, M_0 and \mathbb{C}^n/Γ cannot be regularly symplectomorphic. Unfortunately, for most representations it is practically impossible to determine the affine Poisson algebras $\mathbb{R}[M_0]$ and $\mathbb{R}[\mathbb{C}^n/\Gamma]$ completely. Hence, to tell M_0 apart from \mathbb{C}^n/Γ , one needs to consider quantities that are more amenable to computations. Our basic idea is to observe that, since the Poisson algebra $\mathbb{R}[\mathbb{C}^n/\Gamma]$ has no Casimirs, an isomorphism of Poisson algebras $\mathbb{R}[M_0] \rightarrow \mathbb{R}[\mathbb{C}^n/\Gamma]$ necessarily has to preserve the natural \mathbb{Z} -gradings. Our strategy is to compare the Hilbert series of the \mathbb{Z} -graded algebras $\mathbb{R}[M_0]$ and $\mathbb{R}[\mathbb{C}^n/\Gamma]$.

In this paper, we focus on the computation of the Hilbert series associated to unitary representations of the circle \mathbb{S}^1 . We discover that the Laurent expansions of the Hilbert series of the graded rings $\mathbb{R}[M_0]$ and $\mathbb{R}[\mathbb{C}^n/\Gamma]$ exhibit formal similarities. Namely, in Theorems 5.1 and 6.1, we determine the first four Laurent coefficients of the respective Hilbert series and find that they fulfill the same constraints. As a corollary, we conclude that in order for $\mathbb{R}[M_0]$ and $\mathbb{R}[\mathbb{C}^n/\Gamma]$ to be isomorphic, the weights of the circle representation have to satisfy a certain Diophantine condition (cf. Equation (7.1)). We would like to stress, however, that this condition does not lead us to any new orbifold examples. In a forthcoming paper, we will use the results presented here, in particular Corollary 5.2, to demonstrate that a symplectic quotient M_0 of dimension > 2 of a unitary circle representation V with $V^{\mathbb{S}^1} = \{0\}$ cannot be regularly symplectomorphic to a quotient \mathbb{C}^n/Γ for some finite subgroup $\Gamma \subset U_n$. Note that working over the field \mathbb{C} of complex numbers, there is an analogue of Theorem 5.1 for cotangent lifted representations of the complex circle $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$. Similarly, there is an analogue of Theorem 6.1 for finite subgroups of symplectic group $\mathrm{Sp}_n(\mathbb{C})$. The reader is invited to spell out the details.

A major challenge in the course of proving Theorem 5.1 is to incorporate the cases when the weights degenerate, i.e. two or more weights have the same absolute value. A crucial observation is that in the formulas for the Laurent coefficients in Lemma 5.5, the apparent singularities along the diagonals are actually removable. This leads to the expressions in Theorem 5.1 in terms of Schur polynomials that make perfect sense for the degenerate case. The main purpose of the technology around Theorem 3.1 is to show that these expressions are actually the Laurent coefficients of the Hilbert series of $\mathbb{R}[M_0]$.

Apart from topological conditions (see [7, Subsection 2.2]), we do not know of any *a priori* method in order to tell apart symplectic quotients from quotients of finite subgroups U_n . In fact it seems that orbifold cases appear as accidents, and one has to rely on quantitative, computational means to identify them. Conversely, if one finds a general property of quotients of finite subgroups $\Gamma \subset U_n$, it is expected that the same property holds for symplectic quotients, possibly under suitable topological assumptions. This idea leads to a number of speculations and conjectures. Some of them will be explained in Section 7 and substantiated with sample calculations.

The outline of this paper is as follows. We start by reviewing some background material on invariant theory and symplectic reduction for unitary representations of compact Lie groups in Section 2. Section 3 is devoted to the proof of Theorem 3.1, an expression for Hilbert series of symplectic circle quotients. We present a simple algorithm for the calculation of such a Hilbert series in Section 4. In Section 5, we use Theorem 3.1 to calculate the lowest four Laurent coefficients of the Hilbert series of $\mathbb{R}[M_0]$ (see Theorem 5.1). To compare the Laurent coefficients to the case of finite groups, we calculate the first six Laurent coefficients of the Hilbert series of $\mathbb{R}[\mathbb{C}^n/\Gamma]$ in Section 6. In Section 7 we discuss Diophantine aspects of our findings and speculate about possible generalizations of our results.

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2. BACKGROUND

Suppose $G \rightarrow \mathrm{U}(V)$ is a unitary representation of the compact Lie group G on a finite dimensional complex vector space V with hermitian scalar product $\langle \cdot, \cdot \rangle$. By convention, $\langle \cdot, \cdot \rangle$ is complex antilinear in the first argument. Note that we can make any symplectic representation of G unitary by using an invariant compatible complex structure. Let \bar{V} be the complex conjugate vector space of V . The identity map on V induces a complex antilinear map $-: V \rightarrow \bar{V}$, $v \mapsto \bar{v}$. The complex conjugation $-$ extends to a real structure on the algebra $\mathbb{C}[V \times \bar{V}]$, and the ring of *real regular functions on V* is defined to be the subring of invariants with respect to $-$, i.e. $\mathbb{R}[V] := \mathbb{C}[V \times \bar{V}]^-$. It is isomorphic to the \mathbb{R} -algebra of regular functions on the real vector space $V_{\mathbb{R}}$ underlying V . The group G acts on \bar{V} by $\bar{v} \mapsto (g^{-1})^t \bar{v}$. Letting G act on $V \times \bar{V}$ diagonally, and observing that this action commutes with $-$, we obtain an action of G on $\mathbb{R}[V]$ by \mathbb{R} -algebra automorphisms. This action can be seen as coming from the obvious \mathbb{R} -linear G -action on $V_{\mathbb{R}}$. By the theorem of Hilbert and Weyl, $\mathbb{R}[V]^G$ is a \mathbb{N} -graded Noetherian \mathbb{R} -algebra and we can find a *Hilbert basis*, i.e. complete system of homogeneous polynomial invariants, $\rho_1, \dots, \rho_k \in \mathbb{R}[V]^G$. Note that $v \mapsto \langle v, v \rangle$ is always a quadratic invariant.

Infinitesimally, the data of our representation are encoded by the *moment map* J . This is the regular map from V to the dual space \mathfrak{g}^* of the Lie algebra \mathfrak{g} of G whose image of the point $v \in V$ is determined by

$$J_{\xi}(v) := J(v)(\xi) = \frac{\sqrt{-1}}{2} \langle v, \xi.v \rangle \quad \forall \xi \in \mathfrak{g}.$$

Here $\xi.v = d/dt|_{t=0} \exp(-t\xi).v$ stands for the infinitesimal action of ξ on v . For each $\xi \in \mathfrak{g}$, $J_{\xi}(v) \in \mathbb{R}[V]$ is homogeneous quadratic. Recall that the Kähler form $\omega = \mathrm{Im}\langle \cdot, \cdot \rangle$ is a non-degenerate real-valued two-form on $V_{\mathbb{R}}$. Its inverse, the Poisson tensor Π , is a real-valued two-form on $V_{\mathbb{R}}^*$. Extending it by Leibniz rule we obtain the *Poisson bracket* $\{ \cdot, \cdot \}$, making $\mathbb{R}[V]$ a Poisson algebra over \mathbb{R} . Note that the bracket is homogeneous of degree -2 . The fundamental vector field corresponding

to $\xi \in \mathfrak{g}$ is the Hamiltonian vector field $\{J_\xi, \cdot\}$. Note also that $J : V \rightarrow \mathfrak{g}^*$ is G -equivariant and that $\{J_\xi, J_\eta\} = J_{[\xi, \eta]}$ for $\xi, \eta \in \mathfrak{g}$.

Throughout the paper we will write $Z = J^{-1}(0)$ for the preimage of zero via the moment map. Borrowing terminology from physics we will occasionally refer to Z as the *shell*. Due to the equivariance of J , Z is G -stable. The *symplectic quotient*

$$M_0 = Z/G$$

is the primary object of our investigation. For the ideal of Z in $\mathbb{R}[V]$ we write I_Z , while for the ideal $\langle J_\xi | \xi \in \mathfrak{g} \rangle \subset \mathbb{R}[V]$ we write I_J . If Z is coherent, the inclusion $I_J \subset I_Z$ becomes an equality. For more information on when Z is coherent we refer the reader to [11]. The Poisson algebra $\mathbb{R}[M_0]$ of regular functions on M_0 is $\mathbb{R}[V]^G/I_Z^G$, where $I_Z^G := I_Z \cap \mathbb{R}[V]^G$ is the invariant part of the vanishing ideal.

Recall that, given an \mathbb{N} -graded locally finite dimensional vector space $X = \bigoplus_{i \geq 0} X_i$ over the field \mathbb{K} , there is a generating function

$$\text{Hilb}_{X|\mathbb{K}}(x) = \sum_{i \geq 0} \dim_{\mathbb{K}}(X_i) x^i \in \mathbb{Z}[[x]]$$

for the dimensions of the graded components X_i of X . We will refer to it as the *Hilbert series* of X . It will be convenient to call elements of the ring $\mathbb{R}[V]^G$ *off-shell invariants*. The Hilbert series of the graded ring $\mathbb{R}[V]^G$ will be referred to as the *off-shell Hilbert series* and we write

$$\text{Hilb}_{G:V}^{\text{off}}(x) := \text{Hilb}_{\mathbb{R}[V]^G|\mathbb{R}}(x).$$

By the ring of *on-shell invariants* we mean $\mathbb{R}[V]^G/I_J^G$, where

$$I_J^G := I_J \cap \mathbb{R}[V]^G$$

is the invariant part of I_J . Note that, if G is abelian, I_J^G is actually the ideal generated by the components of the moment map over $\mathbb{R}[V]^G$. By the *on-shell Hilbert series* we mean

$$\text{Hilb}_{G:V}^{\text{on}}(x) := \text{Hilb}_{\mathbb{R}[V]^G/I_J^G|\mathbb{R}}(x).$$

We now turn our attention to the case when G is a torus $\mathbb{T}^\ell = (\mathbb{S}^1)^\ell$. We identify the Lie algebra \mathfrak{g} of $G = \mathbb{T}^\ell$ with \mathbb{R}^n by writing an arbitrary element $(t_1, \dots, t_\ell) \in G = \mathbb{T}^\ell$ in the form $t_i = \exp(2\pi\sqrt{-1}\xi_i)$, for the vector $(\xi^1, \dots, \xi^\ell) \in \mathfrak{g} = \mathbb{R}^n$. We identify V with \mathbb{C}^n by choosing coordinates z_1, \dots, z_n . The data of the representation are encoded by the *weight matrix* $A = (a_{ij}) \in \mathbb{Z}^{\ell \times n}$. Setting $(\eta_1, \dots, \eta_n) := (\xi_1, \dots, \xi_\ell) \cdot A \in \mathbb{R}^n$, the $G = \mathbb{T}^\ell$ -action corresponding to the weight matrix A is given by the formula

$$(2.1) \quad (t_1, \dots, t_\ell) \cdot (z_1, \dots, z_n) = (\exp(2\pi\sqrt{-1}\eta_1)z_1, \dots, \exp(2\pi\sqrt{-1}\eta_n)z_n).$$

The components J_i of the moment map $J = (j_1, \dots, j_n) : \mathbb{C}^n \rightarrow \mathbb{R}^\ell \cong \mathfrak{g}^*$ can also be expressed in terms of the weight matrix,

$$(2.2) \quad J_i(\mathbf{z}, \bar{\mathbf{z}}) = \frac{1}{2} \sum_{j=1}^n a_{ij} z_j \bar{z}_j, \quad i = 1, \dots, \ell.$$

In the case of a torus representation we use the notation $\text{Hilb}_A^{\text{off}}(x) := \text{Hilb}_{G:V}^{\text{off}}(x)$ and $\text{Hilb}_A^{\text{on}}(x) := \text{Hilb}_{G:V}^{\text{on}}(x)$.

Proposition 2.1. *If $A \in \mathbb{Z}^{\ell \times n}$ is a weight matrix of rank $(A) = \ell$, then $\text{Hilb}_A^{\text{on}}(x) = (1 - x^2)^\ell \text{Hilb}_A^{\text{off}}(x)$.*

Proof. This follows from the observation that the moment map cuts out a complete intersection of quadrics of codimension ℓ from $\mathbb{R}[V]^G$ (cf. [10]). \square

In the case of nonabelian G , one cannot expect such a nice relationship between on-and off-shell Hilbert series.

With the exception of Section 7, we will assume for the remainder of the paper that $G = \mathbb{S}^1$. Here, for simplicity, we will talk of a *weight vector* and write $A = (a_1, \dots, a_n) \in \mathbb{Z}^n$. We observe that as the weight matrix of the action of \mathbb{S}^1 on $V \times \overline{V}$ is given by $(a_1, \dots, a_n, -a_1, \dots, -a_n)$, $\text{Hilb}_A^{\text{off}}(x)$ and $\text{Hilb}_A^{\text{on}}(x)$ do not depend on the signs of the weights a_i . For $G = \mathbb{S}^1$ we have $I_J = I_Z$ if and only if not all weights have the same sign (cf. [1, 10]). Note that if all weights do have the same sign, then Z is the origin and the symplectic quotient is a point.

Definition 2.2. We say that the weight vector $A = (a_1, a_2, \dots, a_n) \in \mathbb{Z}^n$ is *generic* if $|a_i| \neq |a_j|$ for all $i \neq j$. Otherwise, A is *degenerate*.

It will sometimes be convenient to assume without loss of generality that our circle action is effective, i.e. that $\gcd(a_1, \dots, a_n) = 1$. Similarly, we occasionally assume that there are no nontrivial invariants, i.e. the weights a_i are nonzero. As the signs of the weights do not affect the Hilbert series, we also occasionally assume that the weights are non-negative.

3. THE HILBERT SERIES OF A SYMPLECTIC CIRCLE QUOTIENT

The aim in this section is to prove the following.

Theorem 3.1. *Let $A = (a_1, \dots, a_n) \in \mathbb{Z}^n$ be a nonzero weight vector for a \mathbb{S}^1 -action on \mathbb{C}^n . The on-shell Hilbert series (as a meromorphic function in x) is given by*

$$(3.1) \quad \text{Hilb}_A^{\text{on}}(x) = \lim_{C \rightarrow A} \sum_{i=1}^n \sum_{\zeta^{a_i}=1} \frac{1}{c_i \prod_{\substack{j=1 \\ j \neq i}}^n (1 - \zeta^{a_j} x^{(c_i+c_j)/c_i})(1 - \zeta^{-a_j} x^{(c_i-c_j)/c_i})}.$$

Here $C = (c_1, \dots, c_n) \in \mathbb{R}^n$ is assumed to be such that $|c_i| \neq |c_j|$ for $i \neq j$.

Note that if A is a *generic* weight vector, then the limit is unnecessary and the formula for the Hilbert series is obtained simply by setting $c_i = a_i$ for $i = 1, \dots, n$; see Proposition 3.2. As well, see Proposition 3.4 for an alternative expression of the Hilbert series in the general case.

Throughout, we will assume that $x \neq 0$ for simplicity, though the extension of our expressions for the Hilbert series to this value can be easily checked.

3.1. The generic case. Let $A = (a_1, \dots, a_n)$ be a generic weight vector. By Molien's formula, the off-shell Hilbert series associated to A is given by the residue

$$\text{Hilb}_A^{\text{off}}(x) = \frac{1}{2\pi i} \int_{z \in \mathbb{S}^1} \frac{dz}{z \prod_{i=1}^n (1 - xz^{a_i})(1 - xz^{-a_i})}, \quad |x| < 1.$$

Rewriting the integrand $F(x, z) = F_A(x, z)$ as

$$F(x, z) = \frac{z^{-1 + \sum_{i=1}^n a_i}}{\prod_{i=1}^n (1 - xz^{a_i})(z^{a_i} - x)},$$

we see that for fixed nonzero $x \in \mathbb{D}^1$, the poles of $F(x, z)$ for $z \in \mathbb{D}^1$ occur when $z^{a_i} = x$. Fix a branch of x^z by choosing a branch of the logarithm in a neighborhood of x and let $R(a_i)$ denote the set of a_i th roots of unity. Then these poles occur at $z = \zeta x^{1/a_i}$ where $i = 1, \dots, n$ and $\zeta \in R(a_i)$.

To compute the residue at a specific pole $z = \zeta_0 x^{1/a_i}$ with $\zeta_0 \in R(a_i)$, we express

$$F(x, z) = \left(\frac{1}{z - \zeta_0 x^{1/a_i}} \right) \frac{z^{a_i-1}}{(1 - xz^{a_i}) \prod_{\substack{\zeta^{a_i}=1 \\ \zeta \neq \zeta_0}} (z - \zeta x^{1/a_i}) \prod_{\substack{j=1 \\ j \neq i}}^n (1 - xz^{a_j})(1 - xz^{-a_j})}$$

to see that this function has a simple pole at $z = \zeta_0 x^{1/a_i}$. It follows that

$$\text{Res}_{z=\zeta_0 x^{1/a_i}} F(x, z) = \frac{1}{a_i(1 - x^2) \prod_{\substack{j=1 \\ j \neq i}}^n (1 - \zeta_0^{a_j} x^{(a_i+a_j)/a_i})(1 - \zeta_0^{a_j} x^{(a_i-a_j)/a_i})},$$

where we simplify using the identity

$$\prod_{\substack{\zeta^{a_i}=1 \\ \zeta \neq \zeta_0}} (\zeta_0 x^{1/a_i} - \zeta x^{1/a_i}) = \zeta_0^{a_i-1} x^{(a_i-1)/a_i} \prod_{\substack{\zeta^{a_i}=1 \\ \zeta \neq 1}} (1 - \zeta) = \zeta_0^{a_i-1} x^{(a_i-1)/a_i} a_i.$$

Summing over each pole of $F(x, z)$ in \mathbb{D}^1 yields the following.

Proposition 3.2. *Let $A = (a_1, \dots, a_n)$ be a generic weight vector. and the on-shell Hilbert series is given by*

$$(3.2) \quad \text{Hilb}_A^{\text{on}}(x) = \sum_{i=1}^n \sum_{\zeta^{a_i}=1} \frac{1}{a_i \prod_{\substack{j=1 \\ j \neq i}}^n (1 - \zeta^{a_j} x^{(a_i+a_j)/a_i})(1 - \zeta^{-a_j} x^{(a_i-a_j)/a_i})}.$$

Remark 3.3. Suppose the representation with weight vector A is not effective so that $\gcd(a_1, \dots, a_n) = a > 1$. Let $b_i = a_i/a$ for each i , and then $B = (b_1, \dots, b_n)$ is the corresponding effective representation. From the perspective of invariant theory, it is obvious that $\text{Hilb}_A(x) = \text{Hilb}_B(x)$. To elucidate this from the perspective of the above computation, we note that

$$\begin{aligned} \text{Hilb}_A(x) &= \sum_{i=1}^n \sum_{\zeta \in R(a_i)} \frac{1}{a_i \prod_{\substack{j=1 \\ j \neq i}}^n (1 - \zeta^{a_j} x^{(a_i+a_j)/a_i})(1 - \zeta^{-a_j} x^{(a_i-a_j)/a_i})} \\ &= \sum_{i=1}^n \sum_{\eta^{b_i}=1} \sum_{\zeta^a=\eta} \frac{1}{ab_i \prod_{\substack{j=1 \\ j \neq i}}^n (1 - (\zeta^a)^{b_j} x^{(b_i+b_j)/b_i})(1 - (\zeta^a)^{-b_j} x^{(b_i-b_j)/b_i})} \\ &= \sum_{i=1}^n \sum_{\eta^{b_i}=1} \frac{1}{b_i \prod_{\substack{j=1 \\ j \neq i}}^n (1 - \eta^{b_j} x^{(b_i+b_j)/b_i})(1 - \eta^{b_j} x^{(b_i-b_j)/b_i})} = \text{Hilb}_B(x). \end{aligned}$$

Hence, there is an a -to-1 correspondence between the poles of $F_A(x, z)$ for $z \in \mathbb{D}^1$ and those of $F_B(x, z)$ for $z \in \mathbb{D}^1$, and the residue of each pole of the latter function is a times the residue of each corresponding pole of the former.

3.2. The degenerate case. Suppose the weight vector A has degeneracies, so that we can express $A = (a_1, \dots, a_1, a_2, \dots, a_2, \dots, a_r, \dots, a_r)$ with each a_i occurring p_i times and $a_i \neq a_j$ for $i \neq j$. The off-shell Hilbert series associated to A is given by

$$\frac{1}{2\pi i} \int_{z \in \mathbb{S}^1} \frac{dz}{z \prod_{i=1}^r (1 - xz^{a_i})^{p_i} (1 - xz^{-a_i})^{p_i}} = \frac{1}{2\pi i} \int_{z \in \mathbb{S}^1} \frac{z^{-1 + \sum_{i=1}^r p_i a_i} dz}{\prod_{i=1}^r (1 - xz^{a_i})^{p_i} (z^{a_i} - x)^{p_i}}$$

with $|x| < 1$. Fixing a branch of x^z , we again have that for fixed nonzero $x \in \mathbb{D}^1$, the poles of the integrand $F_A(x, z) = F(x, z)$ with $z \in \mathbb{D}^1$ occur at $z = \zeta x^{1/a_i}$ for $i = 1, \dots, r$ and $\zeta \in R(a_i)$. For brevity, we set

$$\beta_i(x, z) = (1 - xz^{a_i})^{p_i} (1 - xz^{-a_i})^{p_i}.$$

Now, fix an i and a $\zeta_0 \in R(a_i)$, and then

$$F(x, z) = \left(\frac{1}{(z - \zeta_0 x^{1/a_i})^{p_i}} \right) \frac{z^{p_i a_i - 1}}{(1 - xz^{a_i})^{p_i} \prod_{\substack{\zeta^{a_i}=1 \\ \zeta \neq \zeta_0}} (z - \zeta x^{1/a_i})^{p_i} \prod_{\substack{j=1 \\ j \neq i}}^r \beta_j(x, z)}.$$

Then the residue at $\zeta_0 x^{1/a_i}$ is given by the $(p_i - 1)$ st coefficient of the Taylor series of the expression after the factor in parentheses, which is holomorphic at $z = \zeta_0 x^{1/a_i}$; that is, $\text{Res}_{z=\zeta_0 x^{1/a_i}} F(x, z)$ is equal to

$$\frac{\partial^{p_i-1}}{\partial z^{p_i-1}} \frac{z^{p_i a_i - 1}}{(p_i - 1)! (1 - xz^{a_i})^{p_i} \prod_{\substack{\zeta^{a_i}=1 \\ \zeta \neq \zeta_0}} (z - \zeta x^{1/a_i})^{p_i} \prod_{\substack{j=1 \\ j \neq i}}^r \beta_j(x, z)} \Big|_{z=\zeta_0 x^{1/a_i}}.$$

Summing each residue yields the following.

Proposition 3.4. *Let $A = (a_1, \dots, a_1, a_2, \dots, a_2, \dots, a_r, \dots, a_r)$ with each a_i occurring p_i times and $a_i \neq a_j$ for $i \neq j$. The off-shell Hilbert series is*

$$(3.3) \quad \text{Hilb}_A^{\text{off}}(x) = \sum_{i=1}^r \sum_{\zeta_0^{a_i}=1} \frac{\partial^{p_i-1}}{\partial z^{p_i-1}} G_i(x, z, \zeta_0) \Big|_{z=\zeta_0 x^{1/a_i}},$$

where for each i ,

$$G_i(x, z, \zeta_0) = \frac{z^{p_i a_i - 1}}{(p_i - 1)! (1 - xz^{a_i})^{p_i} \prod_{\substack{\zeta^{a_i}=1 \\ \zeta \neq \zeta_0}} (z - \zeta x^{1/a_i})^{p_i} \prod_{\substack{j=1 \\ j \neq i}}^r \beta_j(x, z)},$$

and the on-shell Hilbert series is

$$(3.4) \quad \text{Hilb}_A^{\text{on}}(x) = \sum_{i=1}^r \sum_{\zeta_0^{a_i}=1} \frac{\partial^{p_i-1}}{\partial z^{p_i-1}} (1 - x^2) G_i(x, z, \zeta_0) \Big|_{z=\zeta_0 x^{1/a_i}}.$$

3.3. Another approach to the degenerate case: Proof of Theorem 3.1.

Although the formulas given in Proposition 3.4 are useful for concrete computations, e.g. to compute the Hilbert series associated to a specific weight vector, we will in the sequel require the more explicit expression for the Hilbert series associated to a degenerate weight vector given in Theorem 3.1. In particular, the direct relationship between the Hilbert series in the generic and degenerate cases is not apparent from Proposition 3.4.

To this end, we approach the computation of the residues in the degenerate case as follows. First express the weight vector as $A = (\mathbf{a}, \mathbf{b}) = (a, \dots, a, b_1, \dots, b_q)$ where a occurs p times and $a \neq b_i$ for each i . To explain this notation, note that we are interested in computing the residues at poles corresponding to the weight a (i.e. poles of the form $z = \zeta x^{1/a}$ in the computation in Subsection 3.2). We make no additional assumptions about the b_i so that, for instance, the weight vector (\mathbf{b}) may itself have degeneracies.

We then consider

$$(3.5) \quad \frac{1}{2\pi i} \int_{z \in \mathbb{S}^1} \frac{dz}{z \prod_{i=1}^p (1 - x_i z^a)(1 - x_i z^{-a}) \prod_{j=1}^q (1 - y_j z^{b_j})(1 - y_j z^{-b_j})},$$

where $|x_i| < 1$ for $i = 1, \dots, p$, the x_1, \dots, x_p are distinct, and $|y_j| < 1$ for $j = 1, \dots, q$. Let $\mathbf{x} = (x_1, \dots, x_p)$ and $\mathbf{y} = (y_1, \dots, y_q)$, and as above let

$$F(\mathbf{x}, \mathbf{y}, z) = F_{(\mathbf{a}, \mathbf{b})}(\mathbf{x}, \mathbf{y}, z) = \frac{1}{z \prod_{i=1}^p (1 - x_i z^a)(1 - x_i z^{-a}) \prod_{j=1}^q (1 - y_j z^{b_j})(1 - y_j z^{-b_j})}$$

denote the integrand. Succinctly, the method here will be to compute the residues corresponding to this integral and then consider the limit as \mathbf{x} approaches a diagonal element $\Delta_p x := (x, \dots, x) \in \mathbb{C}^p$ and $\mathbf{y} \rightarrow \Delta_q x = (x, \dots, x) \in \mathbb{C}^q$.

First, fix a branch of the logarithm near x . We will assume throughout this section that each x_i and y_j is located in a connected ball U about x contained in the domain of this branch, and x^z , x_i^z , and y_j^z will always be defined with respect to this branch. Then the poles of $F(\mathbf{x}, \mathbf{y}, z)$ for $|z| < 1$ occur at $z = \zeta x_i^{1/a}$ for $i = 1, \dots, p$ and ζ an a th root of unity or $z = \eta y_j^{1/b_j}$ for $j = 1, \dots, q$ and η a b_j th root of unity. Hence, the integral in Equation (3.5) is given by

$$\sum_{\zeta^a=1} \sum_{i=1}^p \text{Res}_{z=\zeta x_i^{1/a}} F(\mathbf{x}, \mathbf{y}, z) + \sum_{j=1}^q \sum_{\eta^{b_j}=1} \text{Res}_{z=\eta y_j^{1/b_j}} F(\mathbf{x}, \mathbf{y}, z).$$

It is easy to see that for a specific i and ζ , $\lim_{\mathbf{x} \rightarrow \Delta x} \text{Res}_{z=\zeta x_i^{1/a}} F(\mathbf{x}, \mathbf{y}, z)$ is not defined; see Equation (3.6) below. However, we have the following.

Lemma 3.5. *For a fixed a th root of unity ζ_0 and fixed y_j for $j = 1, \dots, q$,*

$$\sum_{i=1}^p \text{Res}_{z=\zeta_0 x_i^{1/a}} F(\mathbf{x}, \mathbf{y}, z)$$

as a function of (\mathbf{x}, z) admits an analytic continuation to the set

$$\{(\mathbf{x}, z) : |z| < 1, |x_i| < 1, (\zeta_0 x_i^{1/a})^{b_j} \neq y_j \forall i, j\} \subseteq \mathbb{C}^{p+2}.$$

In other words, for suitable values of the y_k , the singularities at points where $x_i = x_j$ for $i \neq j$ are removable.

Proof. For brevity, set

$$\beta_j(y, z) = (1 - yz^{b_j})(1 - yz^{-b_j})$$

for $j = 1, \dots, q$. Fixing a value of i and an a th root of unity ζ_0 and using manipulations similar to those in Subsection 3.1, we express $F(\mathbf{x}, \mathbf{y}, z)$ as

$$\left(\frac{1}{z - \zeta_0 x_i^{1/a}} \right) \frac{z^{a-1}}{(1 - x_i z^a) \prod_{\substack{\zeta^a=1 \\ \zeta \neq \zeta_0}} (z - \zeta x_i^{1/a}) \prod_{\substack{j=1 \\ j \neq i}}^p (1 - x_j z^a)(1 - x_j z^{-a}) \prod_{k=1}^q \beta_k(y_k, z)}.$$

Then when $x_i \neq x_j$ for $i \neq j$, again following the computations in Subsection 3.1,

$$(3.6) \quad \begin{aligned} \text{Res}_{z=\zeta_0 x_i^{1/a}} F(\mathbf{x}, \mathbf{y}, z) \\ = \frac{x_i^{p-1}}{a(1 - x_i^2) \prod_{\substack{j=1 \\ j \neq i}}^p (1 - x_j x_i)(x_i - x_j) \prod_{k=1}^q \beta_k(y_k, \zeta_0 x_i^{1/a})}. \end{aligned}$$

Summing over i and simplifying yields

$$\begin{aligned} \sum_{i=1}^p \text{Res}_{z=\zeta_0 x_i^{1/a}} F(\mathbf{x}, \mathbf{y}, z) \\ = \frac{\sum_{i=1}^p (-1)^{p-i} x_i^{p-1} \prod_{\substack{j=1 \\ j \neq i}}^p (1 - x_j^2) \prod_{\substack{1 \leq j < k \leq p \\ j, k \neq i}} (1 - x_j x_k)(x_k - x_j) \prod_{\substack{j=1 \\ j \neq i}}^p \prod_{k=1}^q \beta_k(y_k, \zeta_0 x_i^{1/a})}{a \prod_{j=1}^p (1 - x_j^2) \prod_{1 \leq j < k \leq p} (1 - x_j x_k)(x_k - x_j) \prod_{j=1}^p \prod_{k=1}^q \beta_k(y_k, \zeta_0 x_j^{1/a})}. \end{aligned}$$

As a polynomial in x_1, \dots, x_p , the numerator of this expression is alternating. To see this, define

$$\begin{aligned} \alpha_i(x_1, \dots, x_p) \\ = (-1)^{p-i} x_i^{p-1} \prod_{\substack{j=1 \\ j \neq i}}^p (1 - x_j^2) \prod_{\substack{1 \leq j < k \leq p \\ j, k \neq i}} (1 - x_j x_k)(x_k - x_j) \prod_{\substack{j=1 \\ j \neq i}}^p \prod_{k=1}^q \beta_k(y_k, \zeta_0 x_i^{1/a}) \end{aligned}$$

so that the numerator is equal to $\sum_{i=1}^p \alpha_i(x_1, \dots, x_p)$. For an odd permutation $\sigma \in \mathcal{S}_p$, an elementary computation yields that for each i ,

$$\alpha_i(x_{\sigma(1)}, \dots, x_{\sigma(p)}) = -\alpha_{\sigma(i)}(x_1, \dots, x_p),$$

from which it follows that the numerator is alternating. Then as every alternating polynomial is divisible by the Vandermonde determinant $\prod_{1 \leq j < k \leq p} (x_j - x_k)$, see [14, Section I.3], it follows that there is a polynomial $S(x_1, \dots, x_p)$ (whose coefficients are functions of $a, b_1, \dots, b_q, y_1, \dots, y_q$, and ζ_0) such that the alternating numerator can be expressed as the product $S(x_1, \dots, x_p) \prod_{1 \leq j < k \leq p} (x_k - x_j)$. Therefore, the singularities at $x_k = x_j$ in the sum of the residues are removable, and $\sum_{i=1}^p \text{Res}_{z=\zeta_0 x_i^{1/a}} F(\mathbf{x}, z)$ admits the continuation

$$(3.7) \quad \frac{S(x_1, \dots, x_p)}{a \prod_{j=1}^p (1 - x_j^2) \prod_{1 \leq j < k \leq p} (1 - x_j x_k) \prod_{j=1}^p \prod_{k=1}^q \beta_k(y_k, \zeta_0 x_j^{1/a})}.$$

This function is clearly analytic on the required domain, completing the proof. \square

Now, using Equation (3.6), we express for a fixed a th root of unity ζ_0

$$\begin{aligned} & \sum_{i=1}^p \operatorname{Res}_{z=\zeta_0 x_i^{1/a}} F(\mathbf{x}, \mathbf{y}, z) \\ &= \sum_{i=1}^p \frac{1}{a(1-x_i^2) \prod_{\substack{j=1 \\ j \neq i}}^p (1-x_j x_i)(1-x_j x_i^{-1}) \prod_{k=1}^q \beta_k(y_k, \zeta_0 x_i^{1/a})}. \end{aligned}$$

Fix x and a branch of x^z in a neighborhood of x . Then for each i , setting $x_i = x^{t_i}$ where each t_i is positive real and $\mathbf{t} = (t_1, \dots, t_p)$, we can express the limit as $(\mathbf{x}, \mathbf{y}) \rightarrow (\Delta_p x, \Delta_q x)$ of $\sum_{i=1}^p \operatorname{Res}_{z=\zeta_0 x_i^{1/a}} F(\mathbf{x}, \mathbf{y}, z)$ as

$$\lim_{\mathbf{t} \rightarrow \Delta_p 1} \sum_{i=1}^p \frac{1}{a(1-x^{2t_i}) \prod_{\substack{j=1 \\ j \neq i}}^p (1-x^{t_j+t_i})(1-x^{t_j-t_i}) \prod_{k=1}^q \beta_k(x, \zeta_0 x^{t_i/a})}$$

Setting $\mathbf{s} = (s_1, \dots, s_p)$, we rewrite this as

$$\lim_{\substack{\mathbf{t} \rightarrow \Delta_p 1 \\ \mathbf{s} \rightarrow \Delta_p 1}} \sum_{i=1}^p \frac{1}{(a/s_i)(1-x^{2t_i/s_i}) \prod_{\substack{j=1 \\ j \neq i}}^p (1-x^{(t_j+t_i)/s_j})(1-x^{(t_j-t_i)/s_j}) \prod_{k=1}^q \beta_k(x, \zeta_0 x^{t_i/a})}$$

so that setting $s_j = t_j$ for $j = 1, \dots, p$ yields

(3.8)

$$\lim_{\mathbf{t} \rightarrow \Delta_p 1} \sum_{i=1}^p \frac{1}{(a/t_i)(1-x^2) \prod_{\substack{j=1 \\ j \neq i}}^p (1-x^{(t_j+t_i)/t_j})(1-x^{(t_j-t_i)/t_j}) \prod_{k=1}^q \beta_k(x, \zeta_0 x^{t_i/a})}$$

where we note that

$$\begin{aligned} (t_j + t_i)/t_j &= (a/t_i + a/t_j)/(a/t_i), \\ (t_j - t_i)/t_j &= (a/t_i - a/t_j)/(a/t_i), \quad \text{and} \\ \beta_k(x, \zeta_0 x^{t_i/a}) &= (1 - \zeta_0^{b_k} x^{(a+t_i b_k)/a})(1 - \zeta_0^{-b_k} x^{(a-t_i b_k)/a}) \\ &= (1 - \zeta_0^{b_k} x^{(a/t_i + b_k)/(a/t_i)})(1 - \zeta_0^{-b_k} x^{(a/t_i - b_k)/(a/t_i)}). \end{aligned}$$

Hence, setting $c_j = a/t_j$ for each $1, \dots, p$ and $\mathbf{c} = (c_1, \dots, c_p)$ yields

(3.9)

$$\lim_{\mathbf{c} \rightarrow \Delta_p a} \sum_{i=1}^p \frac{1}{c_i(1-x^2) \prod_{\substack{j=1 \\ j \neq i}}^p (1-x^{(c_i+c_j)/c_i})(1-x^{(c_i-c_j)/c_i}) \prod_{k=1}^q \beta_k(x, \zeta_0 x^{1/c_i})}$$

where

$$\beta_k(x, \zeta_0 x^{1/c_i}) = (1 - \zeta_0^{b_k} x^{(c_i+b_k)/c_i})(1 - \zeta_0^{-b_k} x^{(c_i-b_k)/c_i}).$$

With this, we have

$$\begin{aligned} & \lim_{(\mathbf{x}, \mathbf{y}) \rightarrow (\Delta_p x, \Delta_q x)} \frac{1}{2\pi i} \int_{z \in \mathbb{S}^1} F(\mathbf{x}, \mathbf{y}, z) dz \\ &= \sum_{\zeta^a=1} \lim_{\mathbf{x} \rightarrow \Delta_p x} \sum_{i=1}^p \text{Res}_{z=\zeta x_i^{1/a}} F(\mathbf{x}, \Delta_q x, z) \\ & \quad + \lim_{(\mathbf{x}, \mathbf{y}) \rightarrow (\Delta_p x, \Delta_q x)} \sum_{j=1}^q \sum_{\eta^{b_j}=1} \text{Res}_{z=\eta y_j^{1/b_j}} F(\mathbf{x}, \mathbf{y}, z), \end{aligned}$$

where the limit of the sum over i exists as above. Switching the roles of a and each value of b_j to apply the above argument to each integer that appears as a weight in A , the limit of the sum over j exists as well, so that the limit of integrals on the left side of the equation exists.

Finally, it remains only to show that

$$(3.10) \quad \lim_{(\mathbf{x}, \mathbf{y}) \rightarrow (\Delta_p x, \Delta_q x)} \frac{1}{2\pi i} \int_{z \in \mathbb{S}^1} F(\mathbf{x}, \mathbf{y}, z) dz = \frac{1}{2\pi i} \int_{z \in \mathbb{S}^1} F(\Delta_p x, \Delta_q x, z) dz = \text{Hilb}_A^{\text{off}}(x).$$

However, note that if $\epsilon = 1 - |x|$ and each x_i and y_j are chosen to have modulus bounded above by $1 - \epsilon/2$, then for $|z| = 1$, $|F(\mathbf{x}, \mathbf{y}, z)|$ is bounded by $2^{p+q}/\epsilon^{p+q}$. Choose sequences $\mathbf{x}_m \rightarrow \Delta_p x$ and $\mathbf{y}_m \rightarrow \Delta_q x$ such that for each m , the coordinates of \mathbf{x}_m and \mathbf{y}_m are all distinct and contained in the ball of radius $1 - \epsilon/2$ about the origin as well as the connected subset U in the domains of each of the branches of $z^{1/a}$ and z^{1/b_j} . Then an application of the Dominated Convergence Theorem yields

$$\lim_{m \rightarrow \infty} \frac{1}{2\pi i} \int_{z \in \mathbb{S}^1} F(\mathbf{x}_m, \mathbf{y}_m, z) dz = \frac{1}{2\pi i} \int_{z \in \mathbb{S}^1} F(\Delta_p x, \Delta_q x, z) dz,$$

so that as the limit on the left side of Equation (3.10) has been shown to exist, Equation (3.10) follows.

With this, applying Equation (3.9) to each degeneracy in the weight vector A completes the proof of Theorem 3.1. Note that for x such that $|x| > 1$, applying the above argument to $1/x$ demonstrates that Equation (3.1) holds on the domain of the rational function $\text{Hilb}_A^{\text{on}}(x)$.

4. AN ALGORITHM FOR COMPUTING THE HILBERT SERIES

In this section, we present a simple algorithm to calculate the on-shell Hilbert series of a unitary circle representation. The algorithm plays an important role in the logic of this work as it provides us with plenty of empirical data. For simplicity, we restrict to the case of a generic weight vector $A = (a_1, \dots, a_n)$. Without loss of generality, we assume that the weights are non-negative.

First let us fix a number $a \in \mathbb{N}$ and introduce an operation $U_a : \mathbb{Q}[[x]] \rightarrow \mathbb{Q}[[x]]$ that assigns to a formal power series $F(x) = \sum_{i \geq 0} F_i x^i$ the series

$$(U_a F)(x) := F_{(a)}(x) := \sum_{i \geq 0} F_{ia} x^i \in \mathbb{Q}[[x]].$$

Lemma 4.1. *If $F(x)$ is a rational power series, then $(U_a F)(x) = F_{(a)}(x)$ is rational as well.*

Proof. Recall the well known fact (cf. [20, Theorem 4.1.1]) that $F(x)$ is rational if and only if there exists a finite collection of polynomials P_1, P_2, \dots, P_k and complex numbers $\mu_1, \mu_2, \dots, \mu_k$ such that $F_n = \sum_{i=1}^k P_i(n) \mu_i^n$. Moreover, if we write $F(x) = P(x)/Q(x)$, then $Q(x)$ factors as $Q(x) = \prod_{i=1}^k (1 - \mu_i x)$. It follows that $F_{(a)}(x)$ is rational, and we can write $F_{(a)}(x) = P_a(x)/Q_a(x)$ with $Q_a(x) = \prod_{i=1}^k (1 - \mu_i^a x)$. \square

It is easy to see that the operation U_a can be understood in terms of averaging over the cyclic groups of order a , i.e.

$$(U_a F)(x) = F_{(a)}(x) = \frac{1}{a} \sum_{\zeta^a=1} F(\zeta \sqrt[a]{x}).$$

This operation has been used before in the context of invariant theory calculations [19].

Let us introduce for each $i = 1, \dots, n$ the function

$$\widetilde{\Phi}_i(x) := \frac{1}{\prod_{j \neq i} (1 - x^{a_i - a_j})(1 - x^{a_i + a_j})}.$$

Note that $\widetilde{\Phi}_i(x)$ is analytic at $x = 0$. Reinterpreting formula (3.2), we find that

$$\text{Hilb}_A^{on}(x) = \sum_{i=1}^n (\widetilde{\Phi}_i)_{(a_i)}(x).$$

We observe that each $(\widetilde{\Phi}_i)(x)$ can be written in the form $P(x)/Q(x)$ where $P(x)$ is a monomial and $Q(x)$ is a product of factors of the form $(1 - x^m)$ with $m > 0$. We use the idea of the proof of Lemma 4.1 to guess the denominator of $(\widetilde{\Phi}_i)_{(a_i)}(x)$. Namely we have to replace each factor according to the rule

$$(4.1) \quad (1 - x^m) \mapsto (1 - x^{\text{lcm}(a_i, m)/a_i})^{\gcd(a_i, m)}.$$

Knowing the Taylor expansion of the rational function $(\widetilde{\Phi}_i)_{(a_i)}(x)$ and its denominator polynomial, the numerator polynomial can be determined by multiplying out. To make this an algorithm, it remains to understand how far we have to calculate the Taylor expansion of $\widetilde{\Phi}_i(x)$. To this end, we use Kempf's bound (see [13, Theorem 4.3]). If we write $\text{Hilb}_A^{off}(x)$ as a fraction $P(x)/Q(x)$, the bound says that $\deg(P) \leq \deg(Q)$. Consequently, we need to determine the Taylor expansion of $\widetilde{\Phi}_i(x)$ merely until degree $a_i(d+2)$ where d is the degree of the denominator of $(\widetilde{\Phi}_i)(x)$ (which coincides with the degree of the denominator of $(\widetilde{\Phi}_i)_{(a_i)}(x)$).

As an example, consider the weight vector $(1, 2, 3)$. We have

$$\begin{aligned} \widetilde{\Phi}_1(x) &= \frac{1}{(1-x^{-1})(1-x^3)(1-x^{-2})(1-x^4)} = \frac{x^3}{(1-x)(1-x^2)(1-x^3)(1-x^4)}, \\ \widetilde{\Phi}_2(x) &= \frac{1}{(1-x)(1-x^3)(1-x^{-1})(1-x^5)} = \frac{-x}{(1-x)^2(1-x^3)(1-x^5)}, \\ \widetilde{\Phi}_3(x) &= \frac{1}{(1-x)(1-x^2)(1-x^4)(1-x^5)}. \end{aligned}$$

Clearly $(\widetilde{\Phi}_1)_{(1)}(x) = \widetilde{\Phi}_1(x)$. Using Equation (4.1), the denominator of $(\widetilde{\Phi}_2)_{(2)}(x)$ is $(1-x)^2(1-x^3)(1-x^5)$ and that of $(\widetilde{\Phi}_3)_{(3)}(x)$ remains $(1-x)(1-x^2)(1-x^4)(1-x^5)$. To compute the numerator of $(\widetilde{\Phi}_2)_{(2)}(x)$, we compute the Taylor series of $\widetilde{\Phi}_2(x)$ to degree $a_2(d+2) = 24$ (where $d = 10$ is the degree of the denominator), apply U_2 ,

and multiply by $(1-x)^2(1-x^3)(1-x^5)$. The numerator of $(\widetilde{\Phi}_2)_{(2)}(x)$ is the sum of the terms of degree at most $d = 10$ in the result,

$$-2z - z^2 - 2z^3 - z^4 - 2z^5.$$

Applying the same process to compute $(\widetilde{\Phi}_3)_{(3)}(x)$ yields the numerator

$$1 + z + 4z^2 + 5z^3 + 5z^4 + 5z^5 + 4z^6 + z^7 + z^8,$$

and hence the Hilbert series is given by

$$\begin{aligned} & (\widetilde{\Phi}_1)_{(1)}(x) + (\widetilde{\Phi}_2)_{(2)}(x) + (\widetilde{\Phi}_3)_{(3)}(x) \\ &= \frac{x^3}{(1-x)(1-x^2)(1-x^3)(1-x^4)} + \frac{-2z - z^2 - 2z^3 - z^4 - 2z^5}{(1-x)^2(1-x^3)(1-x^5)} \\ & \quad + \frac{1 + z + 4z^2 + 5z^3 + 5z^4 + 5z^5 + 4z^6 + z^7 + z^8}{(1-x)(1-x^2)(1-x^4)(1-x^5)} \\ &= \frac{1 + z^2 + 3z^3 + 4z^4 + 4z^5 + 4z^6 + 3z^7 + z^8 + z^{10}}{(1-z^2)(1-z^3)(1-z^4)(1-z^5)}. \end{aligned}$$

We have implemented the algorithm in a Mathematica [16] notebook available at <http://faculty.rhodes.edu/seaton/HilbertSeriesS1.nb>. To give the reader an impression of what can be achieved with the notebook let us consider the weight vector $A = (100, 101, 102) \in \mathbb{Z}^3$. The calculation of the on-shell Hilbert series takes about two hours on a PC. The numerator polynomial is of degree 502 and it takes an A4 page to write it down. The denominator polynomial is $(1-z)(1-z^{101})(1-z^{201})(1-z^{203})$. For $n \leq 5$ and weights bounded by 10, the evaluation takes a few seconds.

5. LAURENT EXPANSION OF $\text{Hilb}_A^{on}(x)$

Throughout this section,

$$\text{Hilb}_A^{on}(x) = \sum_{k=0}^{\infty} \gamma_k(1-x)^{k+2-2n}.$$

denotes the Laurent expansion of the on-shell Hilbert series. We shall write occasionally $\gamma_k(A)$ in order to stress the dependence of γ_k on the data A . Our aim in this section is to prove the following statement.

Theorem 5.1. *Let $A = (a_1, \dots, a_n) \in \mathbb{Z}^n$ be a nonzero weight vector such that $\gcd(A) := \gcd(a_1, \dots, a_n) = 1$. Put $\alpha_j := |a_j|$ for $j = 1, \dots, n$ and write $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$. Let $\alpha_{\widehat{j}} \in \mathbb{N}^{n-1}$ be the vector obtained from α by omitting α_j and set $g_j := \gcd(\alpha_{\widehat{j}})$. Then*

$$\gamma_0(A) = \frac{s_{(n-2, n-2, n-3, \dots, 1, 0)}(\alpha)}{s_{(n-1, n-2, \dots, 1, 0)}(\alpha)},$$

$\gamma_1(A) = 0$, and

$$\begin{aligned} \gamma_2(A) = \gamma_3(A) &= \frac{\gamma_0}{12} + \frac{s_{(n-3, n-3, n-3, n-4, \dots, 1, 0)}(\alpha)}{12 s_{(n-1, n-2, \dots, 1, 0)}(\alpha)} S_{\alpha} \\ & \quad + \sum_{j=1}^n \frac{(g_j^2 - 1) s_{(n-3, n-3, n-4, \dots, 1, 0)}(\alpha_{\widehat{j}})}{12 s_{(n-2, n-3, \dots, 1, 0)}(\alpha_{\widehat{j}})}, \end{aligned}$$

where $S_{\alpha} = \sum_{i=1}^n \alpha_i^2$, and s_{λ} denotes the Schur polynomial associated to the partition λ (see Appendix A).

Corollary 5.2. *Under the assumptions of Theorem 5.1 we have $\gamma_0(A) > 0$ and $\gamma_2(A) = \gamma_3(A) > 0$.*

For the rest of the section, we assume without loss of generality that $a_i > 0$ for each i (hence $A = \alpha$) and $\gcd(A) = 1$; see Section 2, and recall that the last condition corresponds to the representation being effective.

Let $C = (c_1, \dots, c_n)$, where each c_i is a positive real number and $c_i \neq c_j$ for $i \neq j$. Define

$$H_i(C, \zeta) = \frac{1}{c_i \prod_{\substack{j=1 \\ j \neq i}}^n (1 - \zeta^{a_j} x^{(c_i + c_j)/c_i}) (1 - \zeta^{-a_j} x^{(c_i - c_j)/c_i})}$$

for $i = 1, \dots, n$ and

$$H(C) = \sum_{i=1}^n \sum_{\zeta^{a_i}=1} H_i(C, \zeta)$$

so that by Equation (3.1),

$$\text{Hilb}_A^{on}(x) = \lim_{C \rightarrow A} H(C).$$

Let $\gamma_k(i, \zeta, C)$ denote the degree $k + 2 - 2n$ coefficient in the Laurent series of $H_i(C, \zeta)$, and let $\gamma_k(C)$ denote the degree $k + 2 - 2n$ coefficient in the Laurent series of $H(C)$.

In Subsection 5.1, we will compute $\gamma_k(C)$ for $k = 0, 1, 2, 3$. This yields a formula for the coefficients of the Hilbert series in the case that A is generic by evaluating at $C = A$. In Subsection 5.2, we will demonstrate that the Laurent series coefficients are continuous functions of C and use this to derive formulas in the general case.

Remark 5.3. Using the relation

$$\text{Hilb}_A^{on}(x) = (1 - x^2) \text{Hilb}_A^{off}(x)$$

and setting

$$\text{Hilb}_A^{off}(x) = \sum_{k=0}^{\infty} \delta_k (1 - x)^{k+1-2n},$$

it is easy to see that $\gamma_0 = 2\delta_0$ and $\gamma_k = 2\delta_k - \delta_{k-1}$ for $k \geq 1$. Hence, the first four coefficients of the off-shell Laurent series can be computed directly from the first four coefficients of the on-shell Laurent series. In view of Theorem 6.1 we would like to make the following observation.

Corollary 5.4. *As $\delta_1 = \delta_0/2 = \gamma_0/4 > 0$, there cannot exist a regular Poisson diffeomorphism from the full circle quotient V/\mathbb{S}^1 to some quotient \mathbb{C}^{n-1}/Γ of a finite subgroup $\Gamma \subset \text{U}_{n-1}$.*

5.1. The Laurent series coefficients of $H(C)$. Here, we demonstrate the following.

Lemma 5.5. *Let $A = (a_1, \dots, a_n)$ be an effective weight vector with each $a_i > 0$, and let $C = (c_1, \dots, c_n)$ where each c_i is a positive real number and $c_i \neq c_j$ for $i \neq j$.*

For each $j = 1, \dots, n$, let $g_j = \gcd\{a_k : k \neq j\}$. Then the first four coefficients of the Laurent series of

$$\sum_{i=1}^n \sum_{\zeta^{a_i}=1} H_i(C, \zeta).$$

at $x = 1$ are given by

$$(5.1) \quad \gamma_0(C) = \sum_{i=1}^n \frac{c_i^{2n-3}}{\prod_{\substack{j=1 \\ j \neq i}}^n (c_i^2 - c_j^2)}, \quad \gamma_1(C) = 0,$$

$$(5.2) \quad \gamma_2(C) = \gamma_3(C) = -\frac{1}{12} \sum_{i=1}^n \frac{c_i^{2n-5}}{\prod_{\substack{j=1 \\ j \neq i}}^n (c_i^2 - c_j^2)} \sum_{\substack{j=1 \\ j \neq i}}^n c_j^2 + \frac{1}{12} \sum_{j=1}^n \sum_{\substack{i=1 \\ i \neq j}}^n \frac{c_i^{2n-5} (g_j^2 - 1)}{\prod_{\substack{k=1 \\ k \neq i, j}}^n (c_i^2 - c_k^2)}.$$

Proof. Using the Laurent series

$$(5.3) \quad \frac{1}{1-x^t} = \frac{1}{t} (1-x)^{-1} + \frac{t-1}{2t} + \frac{t^2-1}{12t} (1-x) + \frac{t^2-1}{24t} (1-x)^2 + \dots,$$

the Taylor series

$$\frac{1}{1-\zeta x^t} = \frac{1}{1-\zeta} - \frac{t\zeta}{(\zeta-1)^2} (1-x) + \dots,$$

and the Cauchy product formula, it is easy to see that for each i and a_i th root of unity ζ , $H_i(\zeta, C)$ has a pole at $x = 1$ of order $2n - 2(k+1)$ where k is the number of $j \in \{1, \dots, n\}$ such that $\zeta^{a_j} \neq 1$. Hence, if $\zeta^{a_j} \neq 1$ for two or more values of j , then $H_i(\zeta, C)$ has a pole of order strictly smaller than $2n - 6$, implying that it does not contribute to the first four terms of the Laurent series.

As $\gcd(a_1, \dots, a_n) = 1$, it follows that for each a_i th root of unity $\zeta \neq 1$, we have $\zeta^{a_j} \neq 1$ for at least one value of j , and hence $\gamma_0(i, \zeta, C) = \gamma_1(i, \zeta, C) = 0$. So let $\zeta = 1$, and then using Equation (5.3), the first Laurent series coefficients of

$$H_i(1, C) = \frac{1}{c_i \prod_{\substack{j=1 \\ j \neq i}}^n (1 - x^{(c_i+c_j)/c_i}) (1 - x^{(c_i-c_j)/c_i})}$$

for each i are given by

$$(5.4) \quad \begin{aligned} \gamma_0(i, 1, C) &= \frac{c_i^{2n-3}}{\prod_{\substack{j=1 \\ j \neq i}}^n c_i^2 - c_j^2}, & \gamma_1(i, 1, C) &= 0, & \text{and} \\ \gamma_2(i, 1, C) = \gamma_3(i, 1, C) &= \frac{-c_i^{2n-5}}{a_i \prod_{\substack{j=1 \\ j \neq i}}^n c_i^2 - c_j^2} \sum_{\substack{j=1 \\ j \neq i}}^n c_j^2. \end{aligned}$$

Summing over i yields Equation (5.1).

Now, suppose for some fixed j that $\zeta \neq 1$ is an a_k th root of unity for each $k \neq j$. The set of such ζ is precisely the set of nonunit g_j th roots of unity where $g_j := \gcd\{a_k : k \neq j\}$. This set of course may be empty, and in particular is empty if a_j is a degenerate weight.

In this case, $H_i(\zeta, C)$ is given for each $i \neq j$ by

$$\frac{1}{c_i(1 - \zeta^{a_j} x^{(c_i+c_j)/c_i})(1 - \zeta^{-a_j} x^{(c_i-c_j)/c_i}) \prod_{\substack{k=1 \\ k \neq i,j}}^n (1 - x^{(c_i+c_k)/c_i})(1 - x^{(c_i-c_k)/c_i})},$$

whereby one computes that

$$\gamma_2(i, \zeta, C) = -\frac{c_i^{2n-5}}{\prod_{\substack{k=1 \\ k \neq i,j}}^n c_i^2 - c_k^2} \left(\frac{\zeta^{a_j}}{(1 - \zeta^{a_j})^2} \right)$$

and

$$\gamma_3(i, \zeta, C) = \frac{c_i^{2n-6}(c_j - c_i)}{\prod_{\substack{k=1 \\ k \neq i,j}}^n c_i^2 - c_k^2} \left(\frac{\zeta^{a_j}}{(1 - \zeta^{a_j})^3} \right) + \frac{c_i^{2n-6}(c_j + c_i)}{\prod_{\substack{k=1 \\ k \neq i,j}}^n c_i^2 - c_k^2} \left(\frac{\zeta^{2a_j}}{(1 - \zeta^{a_j})^3} \right).$$

To compute the sum over the nonunit g_j th roots of unity, note the following. As $\gcd(a_1, \dots, a_n) = 1$, it must be that a_j is coprime to $g_j = \gcd\{a_k : k \neq j\}$ and hence $\zeta \mapsto \zeta^{a_j}$ is a permutation of the nonunit g_j th roots of unity. Applying Gessel's [8, Corollary 3.3], one obtains

$$\begin{aligned} \sum_{\substack{\zeta^{g_j}=1 \\ \zeta \neq 1}} \frac{\zeta}{(1 - \zeta)^2} &= -\frac{g_j^2 - 1}{12}, & \sum_{\substack{\zeta^{g_j}=1 \\ \zeta \neq 1}} \frac{\zeta}{(1 - \zeta)^3} &= -\frac{g_j^2 - 1}{24}, \\ \text{and} \quad \sum_{\substack{\zeta^{g_j}=1 \\ \zeta \neq 1}} \frac{\zeta^2}{(1 - \zeta)^3} &= \frac{g_j^2 - 1}{24}. \end{aligned}$$

Hence

$$\sum_{\substack{\zeta^{g_j}=1 \\ \zeta \neq 1}} \gamma_2(i, \zeta, C) = \sum_{\substack{\zeta^{g_j}=1 \\ \zeta \neq 1}} \gamma_3(i, \zeta, C) = \frac{c_i^{2n-5}(g_j^2 - 1)}{12 \prod_{\substack{k=1 \\ k \neq i,j}}^n c_i^2 - c_k^2}.$$

Summing over all i and j yields

$$\sum_{i=1}^n \sum_{\substack{j=1 \\ j \neq i}}^n \gamma_2(i, \zeta, C) = \sum_{i=1}^n \sum_{\substack{j=1 \\ j \neq i}}^n \gamma_3(i, \zeta, C) = \sum_{i=1}^n \sum_{\substack{j=1 \\ j \neq i}}^n \frac{c_i^{2n-5}(g_j^2 - 1)}{12 \prod_{\substack{k=1 \\ k \neq i,j}}^n c_i^2 - c_k^2},$$

which may vanish if $g_j = 1$ for each j . Reordering the sum and combining this with Equation (5.4) yields Equation (5.2), completing the proof. \square

Remark 5.6. Computations of higher Laurent series coefficients using the above method become more complicated and lead to sums of the form

$$\sum_{\substack{\zeta^{a_j}=1 \\ \zeta \neq 1}} \frac{\zeta^{a_j+a_k}}{a_i(1 - \zeta^{a_j})^2(1 - \zeta^{a_k})^2}.$$

These are special cases of *Fourier–Dedekind sums*, see [2, Section 4]. In particular, the above sum corresponds to $\sigma_{a_j+a_k}(a_j, a_j, a_k, a_k; a_i)$ in the notation of [2], and the authors are not aware of methods of computing them in general.

5.2. Continuity of the Laurent series coefficients. In the case that A is a generic weight vector, Lemma 5.7 can be used to compute the Laurent series coefficients by evaluating $\gamma_k(A)$ for $k = 0, 1, 2, 3$. If A is degenerate, on the other hand, then the expressions in Equations 5.1 and 5.2 are undefined. In this subsection, we demonstrate that $\lim_{C \rightarrow A} \gamma_k(C)$ is defined for a degenerate weight vector A and yields the Laurent series coefficient γ_k of $\text{Hilb}_A^{\text{on}}(x)$. Specifically, we demonstrate the following.

Lemma 5.7. *With notation as above,*

$$(5.5) \quad \gamma_0(C) = \frac{s_{(n-2, n-2, n-3, \dots, 1, 0)}(C)}{s_{(n-1, n-2, \dots, 1, 0)}(C)},$$

and

$$(5.6) \quad \begin{aligned} \gamma_2(C) = \gamma_3(C) = & \frac{\gamma_0(C)}{12} + \frac{s_{(n-3, n-3, n-3, n-4, \dots, 1, 0)}(C)}{12s_{(n-1, n-2, \dots, 1, 0)}(C)} S_C \\ & + \sum_{j=1}^n \frac{(g_j^2 - 1)s_{(n-3, n-3, n-4, \dots, 1, 0)}(C_j)}{12s_{(n-2, n-3, \dots, 1, 0)}(C_j)}, \end{aligned}$$

where $C_j \in \mathbb{R}^{n-1}$ is given by C with c_j removed.

Proof. Rewrite Equation (5.1) as

$$\gamma_0(C) = \frac{1}{\prod_{1 \leq j < k \leq n} c_j^2 - c_k^2} \sum_{i=1}^n (-1)^{i-1} c_i^{2n-3} \prod_{\substack{1 \leq j < k \leq n \\ j, k \neq i}} c_j^2 - c_k^2,$$

and letting $\rho = (n-2, n-2, n-3, \dots, 1, 0)$, observe that

$$\sum_{i=1}^n (-1)^{i-1} c_i^{2n-3} \prod_{\substack{1 \leq j < k \leq n \\ j, k \neq i}} c_j^2 - c_k^2 = \det(c_i^{\rho_j + n-j})_{1 \leq i, j \leq n}.$$

This can be seen by considering the expansion of the determinant along the first row of the matrix and noting that the minors that appear are Vandermonde determinants in the variables c_j^2 . Hence

$$\gamma_0(C) = \frac{\det(c_i^{\rho_j + n-j})_{1 \leq i, j \leq n}}{\prod_{1 \leq j < k \leq n} (c_j + c_k)(c_j - c_k)} = \frac{s_\rho(C)}{\prod_{1 \leq j < k \leq n} (c_j + c_k)}.$$

A simple computation demonstrates that

$$s_{(n-1, n-2, \dots, 1, 0)}(C) = \prod_{1 \leq j < k \leq n} c_j + c_k,$$

from which Equation (5.5) follows.

Similarly, let $S_C = \sum_{i=1}^n c_i^2$, and use Equation (5.4) to express

$$\begin{aligned} \sum_{i=1}^n \gamma_2(i, 1, C) &= \sum_{i=1}^n \gamma_3(i, 1, C) = \frac{1}{12} \sum_{i=1}^n \frac{c_i^{2n-5}(c_i^2 - S_C)}{\prod_{\substack{j=1 \\ j \neq i}}^n c_j^2 - c_j^2} \\ &= \frac{\gamma_0(C)}{12} - \frac{S_C}{12 \prod_{1 \leq j < k \leq n} c_j^2 - c_k^2} \sum_{i=1}^n (-1)^{i-1} c_i^{2n-5} \prod_{\substack{1 \leq j < k \leq n \\ j, k \neq i}} c_j^2 - c_k^2. \end{aligned}$$

Interpreting the sum as a determinant as above, this time along the second row, yields

$$\sum_{i=1}^n \gamma_2(i, 1, C) = \sum_{i=1}^n \gamma_3(i, 1, C) = \frac{1}{12} \left(\gamma_0(C) + \frac{s_{(n-3, n-3, n-3, n-4, \dots, 1, 0)}(C)}{s_{(n-1, n-2, \dots, 1, 0)}(C)} S_C \right).$$

Finally, applying a computation identical to that for $\gamma_0(C)$ to the final sum in Equation (5.2) (in this case applied to $n-1$ variables) yields that for each j ,

$$\sum_{\substack{i=1 \\ i \neq j}}^n \frac{c_i^{2n-5}}{\prod_{\substack{k=1 \\ k \neq i, j}}^n c_k^2 - c_k^2} = \frac{s_{(n-3, n-3, n-4, \dots, 1, 0)}(C_j)}{s_{(n-2, n-3, \dots, 1, 0)}(C_j)}.$$

Combining the above yields Equation (5.6), completing the proof. \square

Noting that the denominators in Equations (5.5) and (5.6) are always positive when each $c_i > 0$, it is clear that for $k = 0, 1, 2, 3$, $\gamma_k(A)$ is defined for a degenerate weight matrix A . To see that $\lim_{C \rightarrow A} \gamma_k(C)$ is equal to the Laurent series coefficient γ_k of $\text{Hilb}_A^{on}(x)$, note the following. For fixed k ,

$$\gamma_k(C) = \frac{1}{2\pi i} \int_P H(C)(x-1)^{k+3-2n} dx$$

where P can be taken to be a positively oriented circle with center 1 and radius ϵ for sufficiently small ϵ , requiring in particular that each $x \in P$ is contained in the domain of the branch of the logarithm used to define x^z . Let R denote the finite set $R = \{\zeta \neq 1 : \zeta^{\pm a_i} = 1, i = 1, \dots, n\}$. Restricting the values of (c_1, \dots, c_n) to the compact set $\prod_{i=1}^n [a_i - 1/2, a_i + 1/2]$, we may choose ϵ small enough so that for each z with $|z-1| < \epsilon$ and each i and j , $z^{(c_i \pm c_j)/c_i} \notin R$. Then because the singularities of $H(C)$ at $c_i = c_j$ for $i \neq j$ are removable by Lemma 3.5, the integrand $H(C)(x-1)^{k+3-2n}$ is a continuous function of x and the c_i on the set $P \times R$, and hence is bounded. Choosing a sequence $C(k) \rightarrow A$ and applying the Dominated Convergence Theorem as in Subsection 3.3, one obtains

$$\begin{aligned} \lim_{k \rightarrow \infty} \frac{1}{2\pi i} \int_P H(C(k))(x-1)^{k+3-2n} dx &= \frac{1}{2\pi i} \int_P \lim_{k \rightarrow \infty} H(C(k))(x-1)^{k+3-2n} dx \\ &= \frac{1}{2\pi i} \int_P H(A)(x-1)^{k+3-2n} dx. \end{aligned}$$

Theorem 5.1 follows. Moreover, it is an obvious consequence of Lemma A.2 that each Schur polynomial has nonnegative coefficients so that Corollary 5.2 follows immediately.

5.3. The completely degenerate case. Suppose the absolute values of the weights all coincide; we assume without loss of generality that $A = (1, \dots, 1)$. We refer the reader to [5] for a discussion of generators of the invariants. The Hilbert series of the corresponding invariants has a particularly nice form, as we will see below.

We count the invariants directly. It is easy to see that a monomial in z_1, \dots, z_n and $\bar{z}_1, \dots, \bar{z}_n$ is invariant if and only if it is the product of a monomial $p(z_1, \dots, z_n)$ and a monomial $q(\bar{z}_1, \dots, \bar{z}_n)$ such that p and q have the same degree. Using the fact that the Hilbert series of polynomials in n variables is given by

$$\frac{1}{(1-x)^n} = \sum_{k=0}^{\infty} \binom{n+k-1}{k} x^k,$$

it follows that the Hilbert series of invariant polynomials in $z_1, \dots, z_n, \bar{z}_1, \dots, \bar{z}_n$ is given by

$$\text{Hilb}_A^{\text{off}}(x) = \sum_{k=0}^{\infty} \binom{n+k-1}{k}^2 x^k.$$

This is easily seen to be equal to the hypergeometric function ${}_2F_1(n, n, 1, x^2)$, see e.g. [6]. Hence, the on-shell Hilbert series is given by

$$\text{Hilb}_A^{\text{on}}(x) = {}_2F_1(n, n, 1, x^2)(1-x^2).$$

Applying Euler's transformation [6, I. 2.1.4 (23)], we have

$${}_2F_1(n, n, 1, x^2) = {}_2F_1(1-n, 1-n, 1, x^2)(1-x^2)^{2n-1},$$

which along with the definition of ${}_2F_1$ yields

$$\text{Hilb}_A^{\text{on}}(x) = \frac{1}{(1-x^2)^{2n-2}} \sum_{k=0}^{n-1} \binom{n-1}{k}^2 x^{2k}.$$

In particular, a direct computation yields that

$$\gamma_0(A) = \frac{1}{2^{2n}} \binom{2n}{n},$$

which by a simple induction argument is seen to be equal to the $(n-1)$ st coefficient in the Taylor series of $1/\sqrt{1-x}$. That is, letting, for each $n \geq 1$, $A_n \in \mathbb{Z}^n$ be a completely degenerate weight vector, we have

$$\sum_{n=0}^{\infty} \gamma_0(A_{n+1}) x^n = \frac{1}{\sqrt{1-x}}.$$

6. LAURENT EXPANSION IN THE CASE OF A FINITE SUBGROUP OF U_n

The purpose of this section is to prove the following statement about the lowest Laurent coefficients of the ring of real invariants of a finite subgroup of U_n .

Theorem 6.1. *Let Γ be a finite subgroup of U_n . Let $\{g_1, \dots, g_r\}$ be a set of primitive pseudoreflections of Γ (cf. Definition 6.2), let \mathcal{Q} denote the set of $g \in \Gamma$ with eigenvalue 1 of multiplicity $n-2$, and let λ_g and μ_g denote the two non-unit eigenvalues of $g \in \mathcal{Q}$. Then the Laurent expansion of the Hilbert series of the real invariants $\mathbb{R}[\mathbb{C}^n]^\Gamma$ at $x = 1$ is given by*

$$\text{Hilb}_{\mathbb{R}[\mathbb{C}^n]^\Gamma}^{\mathbb{R}}(x) = \sum_{k=0}^{\infty} \gamma_k (1-x)^{k-2n}.$$

where

$$(6.1) \quad \gamma_0 = \frac{1}{|\Gamma|}, \quad \gamma_1 = 0,$$

$$(6.2) \quad \gamma_2 = \gamma_3 = \frac{1}{12|\Gamma|} \sum_{i=1}^r |g_i|^2 - 1,$$

$$(6.3) \quad \gamma_4 = \frac{1}{|\Gamma|} \left(\sum_{i=1}^r \frac{-|g_i|^4 + 50|g_i|^2 - 49}{720} + \sum_{g \in \mathcal{Q}} \frac{\lambda_g \mu_g}{(1 - \lambda_g)^2 (1 - \mu_g)^2} \right),$$

$$(6.4) \quad \gamma_5 = \frac{1}{|\Gamma|} \left(\sum_{i=1}^r \frac{-2|g_i|^4 + 40|g_i|^2 - 38}{720} + \sum_{g \in \mathcal{Q}} \frac{2\lambda_g \mu_g}{(1 - \lambda_g)^2 (1 - \mu_g)^2} \right).$$

In particular,

$$(6.5) \quad 2\gamma_4 - \gamma_5 = \gamma_2.$$

For $g \in \Gamma$, we let g_V denote the corresponding element of $U(V)$ and g_W the corresponding element acting on $W := V \times \bar{V}$. As in the case of reduced spaces by \mathbb{S}^1 -actions, we use the notation

$$\text{Hilb}_{\mathbb{R}[\mathbb{C}^n]^\Gamma} \mathbb{R}(x) = \sum_{k=0}^{\infty} \gamma_k (1-x)^{k-2n}.$$

for the Laurent expansion of the Hilbert series so that γ_k denotes the degree $k-2n$ coefficient. Recall that an element $h \in U(V)$ is called a *pseudoreflection* if V^h has complex codimension 1, or equivalently if h has eigenvalue 1 with multiplicity $n-1$.

The Hilbert series of $\mathbb{R}[V]^\Gamma$ can be computed using Molien's formula (see e.g. [21]), which for Γ finite expresses

$$\text{Hilb}_{\mathbb{R}[V]^\Gamma} \mathbb{R}(x) = \frac{1}{|\Gamma|} \sum_{g \in \Gamma} \frac{1}{\det(\text{id} - g_W^{-1}x)}.$$

Fixing $g \in \Gamma$, choose a basis for V with respect to which g_V is diagonal, say $g_V = \text{diag}(\lambda_1, \dots, \lambda_n)$. The ordered set of complex conjugates of the basis elements yields a basis for \bar{V} , and concatenating these bases yields a basis for W with respect to which $g_W = \text{diag}(\lambda_1, \dots, \lambda_n, \lambda_1^{-1}, \dots, \lambda_n^{-1})$. It follows that the corresponding term in the above sum is given by

$$(6.6) \quad \frac{1}{\det(\text{id} - g_W^{-1}x)} = \prod_{i=1}^n \frac{1}{(1 - \lambda_i x)(1 - \lambda_i^{-1}x)}.$$

For $k = 0, 1, \dots$, we let

$$\prod_{i=1}^n \frac{1}{(1 - \lambda_i x)(1 - \lambda_i^{-1}x)} = \sum_{k=0}^{\infty} \gamma_k(g) (1-x)^{k-2n}$$

denote the Laurent series expansion of the term corresponding to g so that $\gamma_k(g)/|\Gamma|$ denotes the contribution to γ_k of the term in Molien's formulas corresponding to g . That is, for each k ,

$$\gamma_k = \frac{1}{|\Gamma|} \sum_{g \in \Gamma} \gamma_k(g).$$

From Equation (6.6), it is clear that if g_V has eigenvalue 1 with multiplicity m , then $\det(\text{id} - g_W^{-1}x)^{-1}$ has a pole at $x = 1$ of order $2m$. Letting e denote the identity element of Γ , it follows that $\gamma_0(g) = \gamma_1(g) = 0$ for each $g \neq e$, $\gamma_0(e) = 1$, and $\gamma_1(e) = 0$. Hence, we recover the well-known fact that $\gamma_0 = 1/|\Gamma|$ as well as $\gamma_1 = 0$, proving Equation (6.1).

Note that by Equation (6.6), each W^{g_W} has even complex dimension. Therefore, there are no $g \in \Gamma$ such that g_W is a pseudoreflection. With this observation, Equation (6.1) also follow from [18, II. Theorem 3.23].

Let $\mathcal{P} = \{g \in \Gamma : \dim_{\mathbb{C}}(V^{g_V}) = n - 1\}$, i.e. the collection of g such that g_V is a pseudoreflection. We make the following.

Definition 6.2. A subset $\{g_1, \dots, g_r\} \subseteq \mathcal{P}$ will be called a *set of primitive pseudoreflections* (for the representation of Γ on V) if

- for each $g \in \mathcal{P}$, $g = g_i^k$ for some $i \in \{1, \dots, r\}$ and $k \in \mathbb{Z}$, and
- $g_i^k = g_j^\ell \neq e$ implies $i = j$ and $k \cong \ell \pmod{|g_i|}$.

It is easy to see that each Γ admits a set of primitive pseudoreflections, which is not necessarily unique. Given a choice $\{g_1, \dots, g_r\}$ of primitive pseudoreflections, we have

$$\mathcal{P} = \bigcup_{i=1}^r \{g_i^k : 1 \leq k \leq |g_i| - 1\},$$

and for each i , there is a basis for V with respect to which

$$\{g_i^k : 1 \leq k \leq |g_i| - 1\} = \bigcup_{\substack{\zeta^{|g_i|}=1 \\ \zeta \neq 1}} \{\text{diag}(\zeta, 1, \dots, 1) : \zeta^{|g_i|} = 1, \zeta \neq 1\}.$$

For each $g \in \mathcal{P}$ with $g_V = \text{diag}(\lambda_g, 1, \dots, 1)$ (for an appropriately chosen basis for V), we have

$$\frac{1}{\det(\text{id} - g_W^{-1}x)} = \frac{1}{(1 - \lambda_g x)(1 - \lambda_g^{-1}x)} (1 - x)^{2-2n}$$

from which it follows that

$$\gamma_2(g) = \gamma_3(g) = \frac{-\lambda_g}{(1 - \lambda_g)^2}, \quad \gamma_4(g) = \frac{-\lambda_g^3 + \lambda_g^2 - \lambda_g}{(1 - \lambda_g)^4}, \quad \text{and} \quad \gamma_5(g) = \frac{-\lambda_g^3 - \lambda_g}{(1 - \lambda_g)^4}.$$

Noting that $\gamma_2(g) = \gamma_3(g) = 0$ for $g \notin \mathcal{P}$, and choosing a set of primitive pseudoreflections g_1, \dots, g_r , we have

$$\begin{aligned} \gamma_2 = \gamma_3 &= \frac{1}{|\Gamma|} \sum_{g \in \mathcal{P}} \frac{-\lambda_g}{(1 - \lambda_g)^2} \\ &= \frac{1}{|\Gamma|} \sum_{i=1}^r \sum_{\substack{\zeta^{|g_i|}=1 \\ \zeta \neq 1}} \frac{-\zeta}{(1 - \zeta)^2} \\ &= \frac{1}{12|\Gamma|} \sum_{i=1}^r |g_i|^2 - 1, \end{aligned}$$

where the last equation follows from applying Gessel's formula [8, Theorem 4.2] to compute the sum of $\zeta/(1 - t\zeta)^2$ over all g_i th roots of unity ζ , subtracting the term corresponding to $\zeta = 1$, and taking the limit as $t \rightarrow 1$. This demonstrates Equation (6.2).

Similarly, we may compute the sums of $\gamma_4(g)$ and $\gamma_5(g)$ over $g \in \mathcal{P}$ by applying Gessel's [8, Corollary 3.3, (3.7)], which yields that for each i ,

$$\begin{aligned} \sum_{\substack{\zeta^{|g_i|=1} \\ \zeta \neq 1}} \frac{\zeta}{(1-\zeta)^4} &= \sum_{\substack{\zeta^{|g_i|=1} \\ \zeta \neq 1}} \frac{\zeta^3}{(1-\zeta)^4} = \frac{|g_i|^4 - 20|g_i|^2 + 19}{720}, \quad \text{and} \\ \sum_{\substack{\zeta^{|g_i|=1} \\ \zeta \neq 1}} \frac{\zeta^2}{(1-\zeta)^4} &= \frac{|g_i|^4 + 10|g_i|^2 - 11}{720}. \end{aligned}$$

It follows that

$$\begin{aligned} (6.7) \quad \sum_{g \in \mathcal{P}} \gamma_4(g) &= \sum_{i=1}^r \sum_{\substack{\zeta^{|g_i|=1} \\ \zeta \neq 1}} \frac{-\zeta^3 + \zeta^2 - \zeta}{(1-\zeta)^4} \\ &= \sum_{i=1}^r \frac{-|g_i|^4 + 50|g_i|^2 - 49}{720}, \end{aligned}$$

and

$$\begin{aligned} (6.8) \quad \sum_{g \in \mathcal{P}} \gamma_5(g) &= \sum_{i=1}^r \sum_{\substack{\zeta^{|g_i|=1} \\ \zeta \neq 1}} \frac{-\zeta^3 - \zeta}{(1-\zeta)^4} \\ &= \sum_{i=1}^r \frac{-2|g_i|^4 + 40|g_i|^2 - 38}{720}. \end{aligned}$$

To complete the computations of γ_4 and γ_5 , we need to consider the contributions of those $g \in \Gamma$ such that V^{g_V} has complex dimension $n-2$. Let \mathcal{Q} denote the set of such elements, i.e. the set of $g \in \Gamma$ such that g_V has eigenvalue 1 with multiplicity $n-2$. Then for $g \in \mathcal{Q}$ such that $g = \text{diag}(\lambda_g, \mu_g, 1, \dots, 1)$ (for an appropriate basis), we have

$$\frac{1}{\det(\text{id} - g_W^{-1}x)} = \frac{1}{(1 - \lambda_g x)(1 - \lambda_g^{-1}x)(1 - \mu_g x)(1 - \mu_g^{-1}x)} (1-x)^{4-2n}$$

so that

$$\gamma_4(g) = \frac{\lambda_g \mu_g}{(1 - \lambda_g)^2 (1 - \mu_g)^2} \quad \text{and} \quad \gamma_5(g) = \frac{2\lambda_g \mu_g}{(1 - \lambda_g)^2 (1 - \mu_g)^2}.$$

Combining this with Equations (6.7) and (6.8) yields Equations (6.3) and (6.4). As well, a simple computation from these expressions demonstrates the relation $2\gamma_4 - \gamma_5 = \gamma_2$ given in Equation 6.5.

7. EXPERIMENTS AND CONJECTURES

Next, we comment on some number theoretic questions that show up when comparing Theorems 5.1 and 6.1. Obviously, in order for the symplectic quotient of the circle action associated to the weight vector $A = (a_1, \dots, a_n)$ to be regularly symplectomorphic to a finite quotient \mathbb{C}^n/Γ , it is necessary that the following Diofantine condition holds

$$(7.1) \quad \frac{1}{\gamma_0(A)} \in \mathbb{Z}.$$

In a forthcoming paper we show that even if condition (7.1) holds, the symplectic circle quotient cannot be regularly symplectomorphic to a finite quotient (after excluding some trivial special cases). Nonetheless it appears worthwhile to note that weight vectors fulfilling condition (7.1) are rather the exception than the rule. For simplicity, we concentrate on the case $n = 3$ and assume that the non-zero weight vector $A = (a_1, a_2, a_3)$ has nonnegative weights. Then

$$\frac{1}{\gamma_0} = \frac{(a_1 + a_2)(a_1 + a_3)(a_2 + a_3)}{a_1 a_2 + a_1 a_3 + a_2 a_3}.$$

For positive weights, this equals

$$a_1 + a_2 + a_3 - \frac{1}{\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3}},$$

and we conclude that γ_0^{-1} is an integer if and only if the *egyptian fraction* $1/a_1 + 1/a_2 + 1/a_3$ is one over an integer. Moreover, we see that $1/\gamma_0 < a_1 + a_2 + a_3$. It is not difficult to show that if $1/\gamma_0 \in \mathbb{Z}$, the weights a_1, a_2, a_3 cannot be pairwise coprime. Let us introduce the *level* $a_1 + a_2 + a_3$ and the probability to meet a weight vector with positive weights of level $\leq L$ with integral γ_0^{-1} :

$$\begin{aligned} \mathcal{P}(L) &= \frac{|\{(a_1, a_2, a_3) \in \mathbb{N} | a_1, a_2, a_3 > 0; a_1 + a_2 + a_3 \leq L; \gamma_0^{-1} \in \mathbb{Z}\}|}{|\{(a_1, a_2, a_3) \in \mathbb{N} | a_1, a_2, a_3 > 0; a_1 + a_2 + a_3 \leq L\}|} \\ &= |\{(a_1, a_2, a_3) \in \mathbb{N} | a_1, a_2, a_3 > 0; a_1 + a_2 + a_3 \leq L; \gamma_0^{-1} \in \mathbb{Z}\}| / \binom{L}{3}. \end{aligned}$$

The graph of the function $\mathcal{P}(L)$ is depicted in Figure 1, and was obtained by a case-by-case count up to level $L = 3130$. Assuming that, for large L , the probability

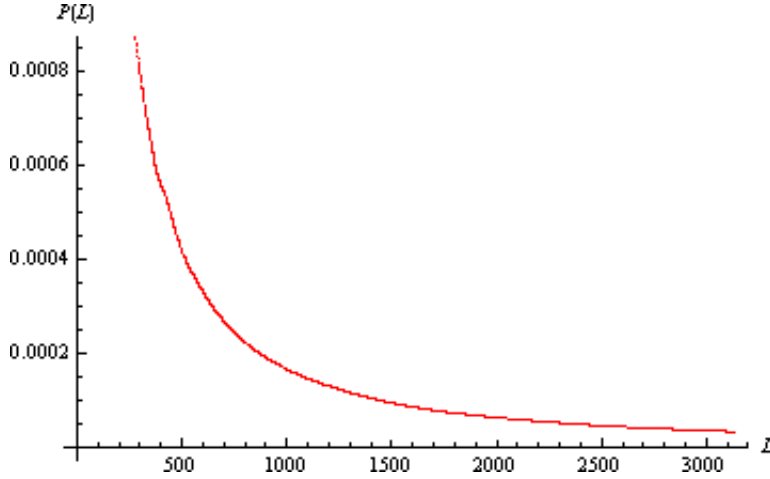


FIGURE 1. A plot of $\mathcal{P}(L)$ for small values of L .

$\mathcal{P}(L)$ follows approximately a power law $\mathcal{P}(L) = \alpha L^{-\beta}$, we estimate the parameters to be $\alpha \approx 2.4105$ and $\beta \approx 1.3898$. Our experiments seem to indicate as well that for $n \geq 4$, the probability $\mathcal{P}(L)$ goes to zero as L goes to infinity. We have no proof for, but believe that, $1/\gamma_0 < \sum_i a_i$ holds also for $n \geq 4$. Due to our lack of expertise in the field, we have to leave the proof of our claims to the interested

number theorist. Note that using Equation (5.1), it is not difficult to see that, for weight vector $A = (a, a, b) \in \mathbb{Z}^3$ with $a, b > 0$, $1/\gamma_0$ is never an integer. The same is true for the totally degenerate case when $n \geq 3$ (see Subsection 5.3).

Inspecting higher order Laurent coefficients (cf. Equations (5.2) and (6.2)), we note that for the symplectic circle quotient with weight vector $A \in \mathbb{Z}^3$ to be regularly symplectomorphic to a \mathbb{C}^2/Γ the condition

$$(7.2) \quad \frac{12\gamma_2(A)}{\gamma_0(A)} \in \mathbb{Z}$$

has to hold as well. This does not lead to any new insights, because our experiments suggest that (7.1) implies (7.2). However occasionally, when $1/\gamma_0$ has few divisors, a circle action can be ruled out using Equation (6.2) as in the following example with $A = (4, 5, 20)$. Here, $\gamma_0 = 1/27$ and $\gamma_2 = 23/162$, so that $12\gamma_2/\gamma_0 = 46$. It is actually impossible to write 46 as the sum of $(m_i^2 - 1)$ such that the $m_i | 27$. Namely, the only possibility is that all $m_i = 3$; but $8 = 3^2 - 1$ does not divide 46.

Finally, we would like to set our results in a more general context, state a conjecture, and provide some empirical evidence for it.

Definition 7.1. We say that a unitary representation $G \rightarrow \mathrm{U}(V)$ of a compact Lie group G has the *bar code of a finite unitary quotient* if in the Laurent expansion $\mathrm{Hilb}_{G;V}^{on}(x) = \sum_{i \geq 0} \gamma_i (1-x)^{i-d}$, $d = \dim M_0$, the coefficients satisfy the following constraints:

- (i) $\gamma_1 = 0$ and $\gamma_2 = \gamma_3$,
- (ii) $2\gamma_4 - \gamma_5 = \gamma_2$.

We do not know of any unitary representation that does not have the bar code of a finite unitary quotient. If it turns out that (possibly under some topological assumptions) every unitary representation has this property, then one should strive to find a conceptual (i.e. symplectic) proof that circumvents the constructive problems of invariant theory. We would like to mention that there might be higher constraints on the γ_i 's of which we are not aware.

In the rest of the section, we describe the representations we have tested to have the bar code of a finite unitary quotient. For circle actions, by Theorem 5.1, the test is necessary merely for property (ii). For $n = 3$ we have systematically checked generic weight vectors with weights ≤ 25 , for $n = 4$ all generic weight vectors with weights ≤ 20 and for $n = 5$ all generic weight vectors with weights ≤ 10 . The example that we have tested with the largest weights is the weight vector $A = (100, 101, 102)$. The totally degenerate case has been tested until $n \leq 100$ (and might be dealt with rigorously). We checked also a couple of other degenerate cases such as $A = (-1, 2, 2)$, $(-2, 1, 1)$, $(-1, 3, 3)$ where we were able to calculate the Hilbert basis and its relations using the software packages Singular [4] and Normaliz [3].

Recall that a weight matrix is called *simplicial* if M_0 is a rational homology manifold (cf. [7]), and note that symplectic orbifolds are necessarily rational homology manifolds. As the signs of the weights of a circle representation do not affect the Hilbert series, when $n \geq 3$, each on-shell Hilbert series of a circle representation correspond to several simplicial weight vectors. For higher dimensional tori, there are more possibilities, and one might wonder whether non-simplicial representations have the bar code of a finite unitary quotient. We tested a handful of simplicial

and non-simplicial cases. To name the simplest simplicial examples,

$$A = \begin{pmatrix} -1 & 0 & 1 & 1 \\ 0 & -1 & 0 & 1 \end{pmatrix} \text{ and } A = \begin{pmatrix} -1 & 0 & 1 & 1 \\ 0 & -1 & 1 & 1 \end{pmatrix},$$

give the same Hilbert series with Laurent expansion

$$\text{Hilb}_A^{on}(x) = \frac{2}{9} \frac{1}{(1-x)^4} + \frac{11}{108} \frac{1}{(1-x)^2} + \frac{11}{108} \frac{1}{1-x} + \frac{49}{432} + \frac{1}{8}(1-x) + \mathcal{O}((1-x)^2).$$

To mention a non-simplicial weight matrix, we have tested

$$A = \begin{pmatrix} -1 & 0 & -1 & 1 & 1 \\ 0 & -1 & 0 & -1 & 1 \end{pmatrix},$$

and the Laurent expansion of the on-shell Hilbert series is given by

$$\text{Hilb}_A^{on}(x) = \frac{19}{144} \frac{1}{(1-x)^6} + \frac{41}{864} \frac{1}{(1-x)^4} + \frac{41}{864} \frac{1}{(1-x)^3} + \frac{407}{6912} \frac{1}{(1-x)^2} + \frac{9}{128} \frac{1}{1-x} + \mathcal{O}(1).$$

In order to address a non-abelian example, we consider the diagonal representation of $G = O_n$ on $V := \mathbb{R}^{4n} = T^*\mathbb{R}^n \times T^*\mathbb{R}^n$, $n = 3$. It has been observed by Huebschmann (see e.g. [12]) that this example appears as a local model for the stratification of the moduli space of flat SU_2 -connection on a Riemann surface of genus 2. For the sake of simplicity, let us forget about the complex structure and work with real coordinates $(\mathbf{q}, \mathbf{p}, \tilde{\mathbf{q}}, \tilde{\mathbf{p}})$. The moment map J of the representation is given by

$$\mathbf{q} \wedge \mathbf{p} + \tilde{\mathbf{q}} \wedge \tilde{\mathbf{p}} : \mathbb{R}^{4n} \rightarrow \wedge^2 \mathbb{R}^n.$$

By the first fundamental theorem of invariant theory, the Hilbert basis is quadratic

$$\begin{aligned} x_1 &= \mathbf{q} \cdot \mathbf{q}, & x_2 &= \mathbf{p} \cdot \mathbf{p}, & x_3 &= \tilde{\mathbf{q}} \cdot \tilde{\mathbf{q}}, & x_4 &= \tilde{\mathbf{p}} \cdot \tilde{\mathbf{p}}, & x_5 &= \mathbf{q} \cdot \mathbf{p}, \\ x_6 &= \mathbf{q} \cdot \tilde{\mathbf{q}}, & x_7 &= \mathbf{q} \cdot \tilde{\mathbf{p}}, & x_8 &= \mathbf{p} \cdot \tilde{\mathbf{q}}, & x_9 &= \mathbf{p} \cdot \tilde{\mathbf{p}}, & x_{10} &= \tilde{\mathbf{q}} \cdot \tilde{\mathbf{p}}. \end{aligned}$$

The inequalities defining V/G have been worked out by Schwarz and Procesi [15]. By the second fundamental theorem of invariant theory, there are no relations; hence $\mathbb{R}[V]^G = \mathbb{R}[x_1, x_2, \dots, x_{10}]$. It is known that $I_J = I_Z$ (see [11]). As the moment map is not invariant, we use elimination theory and Macaulay2 [9] to compute the generators of the ideal $I_J \cap \mathbb{R}[V]^G$ in $\mathbb{R}[V]$, yielding

$$\begin{aligned} &x_2x_7 - x_4x_8 - x_5x_9 + x_9x_{10}, & x_5x_6 + x_3x_7 - x_1x_8 - x_6x_{10}, \\ &x_4x_6 - x_5x_7 + x_1x_9 - x_7x_{10}, & x_2x_6 - x_5x_8 + x_3x_9 - x_8x_{10}, \\ &x_3x_4 - x_7x_8 + x_6x_9 - x_{10}^2, & x_1x_2 - x_5^2 - x_7x_8 + x_6x_9, \\ &x_5^2x_7 - x_1x_4x_8 + x_7^2x_8 - x_1x_5x_9 - x_6x_7x_9 + x_1x_9x_{10}, \\ &x_3x_5x_7 - x_6x_7x_8 - x_1x_3x_9 + x_6^2x_9 + x_3x_7x_{10} - x_6x_{10}^2, \\ &x_3^2x_7^2 - x_1x_3x_7x_8 + x_6^2x_7x_8 + x_1x_3x_6x_9 - x_6^3x_9 - 2x_3x_6x_7x_{10} + x_6^2x_{10}^2. \end{aligned}$$

Experimentation indicates that the result actually does not depend on $n \geq 3$. Using Macaulay2, we are able to compute the minimal free resolution of the ring homomorphism $\mathbb{R}[V]^G \rightarrow \mathbb{R}[V]^G/I_J \cap \mathbb{R}[V]^G = \mathbb{R}[M_0]$. The Betti table turns out

to be

	0	1	2	3	4
0	1
1
2
3	.	6	.	.	.
4	.	.	5	.	.
5
6	.	.	5	.	.
7	.	.	.	6	.
8
9
10	1

Accordingly, the Hilbert series of the ring of on-shell invariants is

$$\begin{aligned} \text{Hilb}_{\mathcal{O}_n:\mathbb{R}^{4n}}^{\text{on}}(x) &= \text{Hilb}_{\mathbb{R}[M_0]|\mathbb{R}}(x) = \frac{1 - 6x^4 + 5x^6 + 5x^8 - 6x^{10} + x^{14}}{(1 - x^2)^{10}} \\ &= \frac{5}{32} \frac{1}{(1 - x)^6} + \frac{11}{128} \frac{1}{(1 - x)^4} + \frac{11}{128} \frac{1}{(1 - x)^3} + \frac{11}{128} \frac{1}{(1 - x)^2} + \frac{11}{128} \frac{1}{(1 - x)^1} + \mathcal{O}(1) \end{aligned}$$

We conclude that this quotient cannot be regularly symplectomorphic to any finite quotient and that the representation has the bar code of a finite unitary quotient. It is conjectured that for $n \geq 3$, all these symplectic quotients are regularly symplectomorphic. We observe that the corresponding GIT quotient is the affine space \mathbb{C}^3 , i.e. it is *smooth*. In the non-abelian case we do not see a way to circumvent the calculation of the Hilbert basis in order to determine the on-shell Hilbert series.

APPENDIX A. SCHUR POLYNOMIALS

Here, we briefly recall the definitions of the Schur polynomials for the benefit of the reader; see [14, Section I.3] and [17, Sections 4.4–6] for more details.

Given $\mu = (\mu_1, \dots, \mu_n) \in \mathbb{Z}^n$, the *alternant* a_μ in the variables $\mathbf{x} = (x_1, \dots, x_n)$ is the alternating polynomial

$$a_\mu(\mathbf{x}) = \det(x_i^{\mu_j})_{1 \leq i, j \leq n}.$$

For $\delta = (n - 1, n - 2, \dots, 0)$, the corresponding alternant $a_\delta(\mathbf{x})$ is called the *Vandermonde determinant* and admits the factorization

$$a_\delta(\mathbf{x}) = \det(x_i^{n-j})_{1 \leq i, j \leq n} = \prod_{1 \leq j < k \leq n} (x_j - x_k).$$

In particular, it follows that $a_\delta(\mathbf{x})$ divides every alternating polynomial in \mathbf{x} , and hence any alternant $a_\mu(\mathbf{x})$.

Definition A.1. Let $\rho = (\rho_1, \dots, \rho_n) \in \mathbb{Z}^n$ with $\rho_1 \geq \rho_2 \geq \dots \geq \rho_n$. The *Schur polynomial* associated to ρ in the variables x_1, \dots, x_n is the symmetric polynomial defined by

$$s_\rho(\mathbf{x}) = \frac{a_{\delta+\rho}(\mathbf{x})}{a_\delta(\mathbf{x})} = \frac{\det(x_i^{\rho_j+n-j})_{1 \leq i, j \leq n}}{\det(x_i^{n-j})_{1 \leq i, j \leq n}}.$$

Alternatively, the Schur polynomials can be characterized as follows. Recall that a *generalized tableau* T of shape $\mu = (\mu_1, \dots, \mu_n) \in \mathbb{Z}^n$ is a left-justified array with n rows of lengths μ_i having positive integer entries $T_{i,j}$ with $1 \leq i \leq n$ and $1 \leq j \leq \mu_i$. A generalized tableau is *semistandard* if its rows nondecreasing and its columns are increasing. Given a generalized tableau T of shape μ , let

$$\mathbf{x}^T = \prod_{(i,j) \in \mu} x_{T_{i,j}}.$$

That is, \mathbf{x}^T is given by the product of all x_t where t ranges over the entries of T .

Lemma A.2. *Let $\rho = (\rho_1, \dots, \rho_n) \in \mathbb{Z}^n$ with $\rho_1 \geq \rho_2 \geq \dots \geq \rho_n$. Then*

$$s_\lambda(\mathbf{x}) = \sum_T \mathbf{x}^T$$

where the sum is over all semistandard λ -tableaux T whose entries are elements of $\{1, \dots, n\}$.

For the proof, see [17, Corollary 4.6.2], and note that Sagan uses this description as the definition of the Schur polynomials. In particular, note that is obvious from this definition that $s_\lambda(\mathbf{x}) > 0$ for any λ when $x_i > 0$, proving Corollary 5.2.

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